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ON THE BENDING OF AN ELASTIC CYLINDER BY RIGID BANDS

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S. I. LITOVCHENKO and B. M. NULLER
(Leningrad)

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We construct a system of piecewise-homogeneous solutions of the asymmetric problem of the theory of elasticity for an infinite cylinder whose surface is partly free of stresses and the remaining part is under the conditions of sliding support. In particular, we obtain the exact solution of the problem of flexure of an infinite cylinder, acted upon by a moment and a force and embedded without friction in a semi-infinite absolutely rigid cylindrical band. The system is used for the solution of four periodic problems regarding the flexure of a cylinder by finite bands and also for the examination of those limiting cases in which the lengths of the adjacent bands or the distances between them are infinitely large. All these problems are reduced to the Poincaré-Koch normal systems.

In applications, the most important cases are those nonaxisymmetric contact problems for the elastic cylinder in which the interior boundaries of the bands represent surfaces of revolution and the bending of the cylinder is due only to the rotation and relative displacements of the bands (for example, bearings) or to its own weight. In the cylindrical coordinates $z,\ r,\ \varphi$, the elastic displacements are then decomposed into axisymmetric ones [1] and those proportional to $\cos\varphi$. The latter case is studied in this paper.

1. We consider two particular elements of the system of piecewise-homogeneous solutions, satisfying at the boundary $r = 1$ the conditions

$$\tau_{r\varphi} = \tau_{rz} = 0, \qquad -\infty < z < \infty \tag{1.1}$$

$$u = 0, \qquad z \geqslant 0 \tag{1.2}$$

$$\sigma_r = 0, \qquad z < 0 \tag{1.3}$$

$$\sigma_r = O(z^\alpha), \qquad z \to +0,\ \alpha > -1 \tag{1.4}$$

and determining the bending of the cylinder by a moment and a transverse force. We note that they cannot be constructed by analogy with the similar elements od the axisymmetric problem [1]. The corresponding solutions in the neighborhood of the separation line have an unbounded stress energy, namely, in spite of the condition (1.4), $\sigma_r = O(z^{-5/2})$ for $r = 1,\ z \to +0$.

We write the solution of the Papkovich-Neuber form

$$\begin{aligned} u_1 &= u \cos^{-1}\varphi = 4(1-\nu) B_r - \partial/\partial r\,(rB_r + B_0) \\ u_2 &= w \cos^{-1}\varphi = -\partial/\partial z\,(rB_r + B_0) \\ u_3 &= v \sin^{-1}\varphi = 4(1-\nu) B_\varphi - \partial/\partial\varphi\,(B_r + r^{-1}B_0) \end{aligned} \tag{1.5}$$

$$\Delta B_r - 2r^{-2}B_\varphi - r^{-2}B_r = 0, \qquad \Delta B_\varphi - 2r^{-2}B_r - r^{-2}B_\varphi = 0 \tag{1.6}$$

$$\Delta B_0 = 0,\ \Delta = \partial^2/\partial r^2 + r^{-1}\,\partial/\partial r - r^{-2} + \partial^2/\partial z^2$$

and we subject the relations (1.6) to the two-sided Laplace transform. Evaluating the functions B_r, B_φ and B_0, by the inversion formulas, we insert them into (1.5). By virtue of (1.1) we obtain

$$u_\varkappa(z, r) = \frac{1}{2\pi i}\int_L C(p) f_\varkappa(p, r) e^{pz}\, dp \tag{1.7}$$

$$f_1(p, r) = [J_1'(pr) - 4(1-\nu) p^{-1}r^{-1}J_1(pr)]\Delta_1(p) - [4(1-\nu) J_1'(pr) + (pr - p^{-1}r^{-1}) J_1(pr)]\,\Delta_2(p) - pJ_1'(pr)\,\Delta_3(p)$$

$$f_2(p, r) = J_1(pr)\,\Delta_1(p) + prJ_1'(pr)\,\Delta_2(p) - pJ_1(pr)\,\Delta_3(p)$$

$$f_3(p, r) = [4(1-\nu) p^{-1}r^{-1} J_1(pr) - J_1'(pr)]\,\Delta_2(p) + [4(1-\nu) J_1'(pr) - p^{-1} r^{-1} J_1(pr)]\,\Delta_1(p) + r^{-1} J_1(pr)\,\Delta_3(p)$$

$$f_4(p, r) = 2G\,\{[p\,J_1'(pr) - 2(1-\nu) r^{-1} J_1(pr)]\,\Delta_1(p) + [(r^{-1} - p^2 r)\,J_1(pr) - 2(1-\nu)\,p\,J_1'(pr)]\,\Delta_2(p) - p^2\,J_1'(pr)\,\Delta_3(p)\}$$

$$f_5(p, r) = 2G\,\{pJ_1(p, r)\,\Delta_1(p) + [p^2 r\,J_1'(pr) + 2\nu p\,J_1(pr)]\,\Delta_2(p) - p^2 J_1(pr)\,\Delta_3(p)\}$$

$$f_6(p, r) = 2G\,\{[2(1-\nu)\,pJ_1'(pr) - r^{-1} J_1(pr)]\,\Delta_1(p) + [2(1-\nu) r^{-1} J_1(pr) - pJ_1'(pr)]\,\Delta_2(p) + pr^{-1} J_1(pr)\Delta_3(p)\}$$

$$f_7(p, r) = 2G\,\{[[(5-4\nu)\,p^{-1}r^{-2} - p]\,J_1(pr) - (5-4\nu)\,r^{-1}J_1'(pr)]\,\Delta_1(p) + [[(3-2\nu)\,p - (5-4\nu)\,p^{-1}r^{-2}]\,J_1(pr) + [(5-4\nu)\,r^{-1} - p^2 r]\,J_1'(pr)] \times \Delta_2(p) + [(p^2 - r^{-1})\,J_1(pr) + pr^{-1}\,J_1'(pr)]\,\Delta_3(p)\}$$

$$f_8(p, r) = 2G\,\{[[(5-4\nu)p^{-1}r^{-2} - 2(1-\nu)\,p]\,J_1(pr) - (5-4\nu)r^{-1}J_1'(pr)]\Delta_1(p) + [(5-4\nu)\,r^{-1}\,J_1'(pr) + [p - (5-4\nu)\,p^{-1}r^{-2}]\,J_1(pr)]\Delta_2(p) + [pr^{-1}\,J_1'(pr) - r^{-2}\,J_1(pr)]\Delta_3(p)\}$$

$$f_9(p, r) = 2G\,\{(5-4\nu)\,[r^{-1}\,J_1'(pr) - p^{-1}r^{-2}J_1(pr)]\,\Delta_1(p) + [[(5-4\nu)\,p^{-1}r^{-2} - (1-2\nu)\,p]\,J_1(pr) - (5-4\nu)r^{-1}\,J_1'(pr)]\,\Delta_2(p) + [r^{-2}\,J_1(pr) - pr^{-1}J_1'(pr)]\Delta_3(p)\}$$

$$\Delta_1(p) = (p^2 - 1)\,J_1^2(p) - 2(1-\nu)\,pJ_1(p)J_1'(p) + (3-2\nu)\,p^2\,J_1'^2(p)$$

$$\Delta_2(p) = 2(1-\nu)\,[(p^3 - p)\,J_1(p)\,J_1'(p) - J_1^2(p) + 2p^2\,J_1'^2(p)]$$

$$\Delta_3(p) = [(1-2\nu)(5-4\nu)\,p^{-1} + (5-4\nu)\,p - 2(1-\nu)p^3]\,J_1^2(p) - 4(1-\nu)(2-\nu)\,p^2\,J_1(p)\,J_1'(p) - (1-2\nu)(5-4\nu)\,pJ_1'^2(p)$$

$$u_4 = \cos^{-1}\varphi\,\tau_{rz}, \qquad u_5 = \cos^{-1}\varphi\sigma_z, \qquad u_6 = \sin^{-1}\varphi\tau_{\varphi z} \tag{1.8}$$

$$u_7 = \cos^{-1}\varphi\sigma_r, \qquad u_8 = \sin^{-1}\varphi\tau_{r\varphi}, \qquad u_9 = \cos^{-1}\varphi\sigma_\varphi$$

Here the contour L is chosen at the left of the imaginary axis, $J_n(pr)$ is the nth order Bessel function of the first kind and the prime denotes differentiation with respect to the argument.

According to the conditions (1.2) and (1.3), the function $C(p)$ satisfies the equations

$$u^-(p) = C(p)\,f_1(p, 1), \quad \sigma^+(p) = C(p)\,f_7(p, 1) \tag{1.9}$$

where

$$u^-(p) = \int_{-\infty}^{0} u_1(z, 1)\, e^{-pz}\, dz, \qquad \sigma^+(p) = \int_{0}^{\infty} u_7(z, 1)\, e^{-pz}\, dz$$

Eliminating it, we obtain the homogeneous Wiener-Hopf equation

$$\sigma^+(p) = K(p)\, u^-(p), \quad K(p) = D_2(p)\, D_1^{-1}(p) \tag{1.10}$$

$$D_1(p) = f_1(p, 1) = -2(1-\nu)\{(p - p^{-1}) J_1^3(p) - 4(1-\nu) J_1^2(p) J_1'(p) + [2(1-\nu)p^3 + p] J_1(p) J_1'^2(p) + 4(1-\nu) p^2 J_1'^3(p)\}$$

$$D_2(p) = -4G(1-\nu)\{(p^5 - 3p^3 + 2\nu p) J_1^3(p) + 2p^2(p^2 - 1 + 2\nu) J_1^2(p) J_1'(p) + p^3(p^2 - 4 + 2\nu) J_1(p) J_1'^2(p) + 2p^4 J_1'^3(p)\}$$

At the point $p = 0$ the function $D_1(p)$ has a zero of multiplicity four and $D_2(p)$ has a zero of multiplicity eight. We show that these functions have no other pure imaginary zeros, generating nontrivial homogeneous solutions.

The problem of the absence of pure imaginary zeros for the characteristic functions of the type $D_2(p)$ has been studied already in [2], Sect. 4 at the proof of Saint-Venant's principle for a cylindrical domain of an arbitrary cross section. However, multiple zeros are not considered in [2] and the condition of the absence of stresses at the boundary of the cylinder is used in an essential manner. The analyses of the functions $D_1(p)$ and $D_2(p)$ given below can be extended to all the other cases of homogeneous boundary conditions, for example to the case of the contact of the cylinder with an elastic shell.

Thus, we assume that $p = i\beta$ is an n-fold imaginary zero of the function $D_1(p)$ or $D_2(p)$. Then $p = -i\beta$ is a zero of the same multiplicity and the corresponding chain of homogeneous solutions $u_{\varkappa k}$, satisfying for $r = 1$, $-\infty < z < \infty$, the conditions

$$\tau_{r\varphi} = \tau_{rz} = u = 0 \quad \text{or} \quad \tau_{r\varphi} = \tau_{rz} = \sigma_r = 0$$

is represented in the form

$$u_{\varkappa k} = \operatorname{Re}\{\partial^k / \partial p^k [f_\varkappa(p, r)\, e^{pz}]\}_{p=i\beta} \qquad (0 \leqslant k \leqslant n-1)$$

According to (1.7), the functions $f_1(i\beta, r)$, $f_3(i\beta, r)$, $f_5(i\beta, r)$ are even with respect to β, and therefore real, while $f_2(i\beta, r)$, $f_4(i\beta, r)$, $f_6(i\beta, r)$ are odd and imaginary; hence it follows that the functions u_{10}, u_{30}, u_{50} are proportional to $\cos\beta z$ and equal to zero at the ends of the cylinder $r = 1$, $z = \pm {}^1/_2\, \pi\beta^{-1}$. By Kirchhoff's uniqueness theorem, we have $u_{\varkappa 0} \equiv 0$ and so $f_\varkappa(i\beta, r) \equiv 0$. But then $u_{\varkappa 1} = \operatorname{Re}[e^{i\beta z} f_\varkappa^*(i\beta, r)]$, where the asterisk denotes derivative with respect to p. Since by a single differentiation with respect to p the parity changes to the opposite one, the functions u_{21}, u_{41}, u_{61} are proportional to $\cos\beta z$, $u_{\varkappa 1} \equiv 0$ and, consequently, $f_\varkappa^*(i\beta, r) \equiv 0$. Continuing these arguments, we obtain $u_{\varkappa k} \equiv 0$ for all k, QED.

Following [3], we can prove that the larger zeros a_{ks} and b_{ks} $(s = 1, 2, 3)$ of the functions $D_1(p)$ and $D_2(p)$ in the right halfplane $\operatorname{Re} p > 0$ $(k = 1, 2, \ldots)$ lie close to the zeros of the functions $\sin^2(p - {}^3/_4\, \pi)\cos(p - {}^3/_4\, \pi)$ and $(2p + \cos 2p)\cos(p - {}^3/_4\, \pi)$, respectively; more exactly we have the asymptotic formulas

$$\begin{gathered} a_{k1} = a_{k2} + O(k^{-1}\ln k) = k\pi - {}^1/_4\, \pi + O(k^{-1}\ln k) \\ a_{k3} = b_{k3} + O(k^{-1}\ln k) = k\pi + {}^1/_4 \pi + O(k^{-1}\ln k) \\ b_{k1} = \bar{b}_{k2} = k\pi + {}^1/_2 \pi + i/2 \ln[2(2k+1)\pi] + O(k^{-1}\ln k) \end{gathered} \tag{1.11}$$

The functions $D_1(p)$ and $D_2(p)$ are even, therefore the numbers $a_{-ks} = -a_{ks}$ and $b_{-ks} = -b_{ks}$ are also their zeros. The functions $D_1(p)$ and $D_2(p)$ do not have other large zeros. The proof of the last assertion is carried out in the same way as in [3], except that the contours in Rouché's theorem have to be constructed in an other way. For the investigation of a_{ks} the contour consists of the segments

$$|\operatorname{Im} p| \leqslant {}^1/_6 \ln k\pi, \ \operatorname{Re} p = \pm (k + {}^3/_4)\pi$$
$$|\operatorname{Re} p| \leqslant (k + {}^3/_4)\pi, \quad \operatorname{Im} p = \pm {}^1/_6 \ln k\pi$$

while for the investigation of b_{ks} of the segments

$$|\operatorname{Im} p| \leqslant {}^1/_2 \ln[4(2k+1)\pi], \ \operatorname{Re} p = \pm (k + {}^3/_4)\pi$$
$$|\operatorname{Re} p| \leqslant (k + {}^3/_4)\pi, \quad \operatorname{Im} p = \pm {}^1/_2 \ln[4(2k+1)\pi]$$

The estimates (1.11) allow us to justify the convergence and the algebraic growth of the infinite products obtained as a result of the factorization

$$K(p) = K_0 p^4 K^-(p)[K^+(p)]^{-1}, \quad K_0 = {}^1/_2 G(1+\nu)$$

$$K^+(p) = [K^-(-p)]^{-1} = \prod_{n=1}^{\infty} \prod_{s=1}^{3} (1 + p a_{ns}^{-1})(1 + p b_{ns}^{-1})^{-1} \tag{1.12}$$

$$K^+(p) \sim \sqrt{{}^1/_2 (1-\nu^2) p^3}, \qquad p \to \infty$$

By virtue of (1.4), (1.12) and the generalized Liouville theorem [4], the solution of Eq. (1.10) has the form

$$\sigma^+(p) = (A_{01} p + A_{02})[K^+(p)]^{-1} \tag{1.13}$$

The elastic displacements and the stresses (1.7) are expressed by the formulas

$$u_\varkappa^{0s} = \frac{1}{2\pi i} \int_L A_{0s} p^{2-s} [K^+(p) D_2(p)]^{-1} f_\varkappa(p, r) e^{pz} dp \quad (s = 1, 2) \tag{1.14}$$

These solutions determine, in particular, the state of stress of an infinite cylinder which is fixed in the band $r = 1$, $0 \leqslant z < \infty$ and is bent by the moment M, applied at $z = -\infty$ (Fig. (*1*)), or by the tangential load, statically equivalent to the transverse force P and applied to the end of the cylinder $z \geqslant -\mu$ for $\mu \gg 1$ (*2*). In both cases the constants A_{0s} are computed from the equilibrium conditions, imposed on the residues of the integrands (1.14) at the poles of order four: the remaining terms in the residue expansions are self-balanced. The exact solution of Problem 1 is given by the formulas (1.14) for

$$s = 1, \ A_{01} = \pi^{-1} M \tag{1.15}$$

The solution of Problem 2 is expressed by the sum $u_\varkappa^{01} + u_\varkappa^{02}$, where

$$A_{01} = \pi^{-1}[\mu + K^{+*}(0)] P, \ A_{02} = \pi^{-1} P \tag{1.16}$$

Due to the absence of the imaginary b_{ks}, it holds in a domain sufficiently far from the ends, where the exponentially decreasing homogeneous solutions are damped.

In both problems the stress concentration under the boundaries of the bands, by virtue of (1.12), (1.13) and the known estimates [4], having the form

$$\sigma_r \sim \frac{2M \cos\varphi}{\pi \sqrt{2(1-\nu^2)\pi z}}, \quad \sigma_r \sim \frac{2P\mu \cos\varphi}{\pi \sqrt{2(1-\nu^2)\pi z}} \quad (r = 1, z \to +0) \tag{1.17}$$

leads to a separation of the cylinder from half of the band edge. Comparing (1.17) with the axisymmetric stresses under the band $r = 1 - \delta,\ 0 \leqslant z < \infty$, placed on the cylinder $r \leqslant 1,\ -\infty < z < \infty$ [1]

$$\sigma_r \sim -\delta G\,(2 + 2\nu)^{1/2}\,(1 - \nu)^{-1}\pi^{-1/2}\,z^{-1/2} \quad (r = 1,\ z \to +0)$$

we obtain the relation

$$\delta \geqslant |M + P\,\mu\cos\varphi_1|\,(1 - \nu)^{1/2}\,[\pi G\,(1 + \nu)]^{-1} \tag{1.18}$$

in which case under the entire edge of the bands we have compressive stresses. Here φ_1 is the angle between the planes of action of M and P.

As the third fundamental element of the system we choose the rotation of the cylinder as a rigid body

$$u_1^{03} = A_{03}\,z, \quad u_2^{03} = A_{03}\,r, \quad u_3^{03} = -A_{03}\,z \tag{1.19}$$

The subsystems of the piecewise-homogeneous solutions, satisfying the conditions (1.1)-(1.4) and having an exponential growth for $z \to \pm\infty$, are written without proof, similar to the axisymmetric case [1]

$$u_\varkappa^{ks} = A_{ks}\Big[f_\varkappa(a_{ks}, r)\,e^{a_{ks}z} + \frac{1}{2\pi i}\int_L g_k(p)\,f_\varkappa(p, r)\,e^{pz}\,dp\Big] \quad (k = 1, 2, \ldots) \tag{1.20}$$

$$u_\varkappa^{ks} = A_{ks}\Big[f_\varkappa(b_{ks}, r)\,e^{b_{ks}z} + \frac{1}{2\pi i}\int_L h_k(p)\,f_\varkappa(p, r)\,e^{pz}\,dp\Big] \quad (k = -1, -2, \ldots)$$

Here the relations (1.5) and (1.8) between the functions $u_\varkappa^{ks}$ and the real displacements and stresses are preserved; if $b_{k2} = \bar{b}_{k1}$, then

$$A_{k2} = \bar{A}_{k1}$$

$$g_k(p) = -\frac{b_{ks}^2 K_0 K^-(-b_{ks})\,D_1(b_{ks})\,p^2}{(p + b_{ks})\,K^+(p)\,D_2(p)}, \qquad h_k(p) = \frac{K^+(a_{ks})\,D_2(a_{ks})\,p^2}{a_{ks}^2\,(p - a_{ks})\,K^+(p)\,D_2(p)}$$

Expanding the integrals (1.20) into residue series, we can see that every element of these subsystems is balanced for $z < 0$ (this is done as in [5], except that we make use of the two constants occurring in (1.13)), and satisfies the mixed boundary conditions (1.1) - (1.3) and the equations of the theory of elasticity.

With the aid of the systems (1.14), (1.19), (1.20) we can solve the different mixed problems for the semi-infinite and finite cylinders, reducing them to infinite systems of algebraic equations having a normal determinant in the case of arbitrary end conditions. If at the end faces of the cylinder we are given combinations of w, $\tau_{\varphi z}$, τ_{rz} or of u, v, σ_z, or if junction conditions are formulated (as in the case of the bending of an infinite cylinder by two finite bands), then the free terms in the systems of algebraic equations can be computed exactly. In other cases, for example the bending of a finite cylinder with unloaded ends by bands, the free terms are computed by the method of least squares.

2. We consider bending problems for an infinite cylinder on which absolutely rigid cylindrical bands are placed periodically. Assume that the radii of the cylinder and of the bands are equal to unity, the distance between neighboring bands is 2μ, their length is 2λ, the semiplane $\varphi = 0$ coincides with the plane of the figure below the axis of the cylinder, there are no friction forces on the contact surfaces, and exrernal forces are applied only to the bands.

The coordinate plane $z = 0$ is taken through the left end of a middle band. Then the boundary conditions on the lateral surface of the cylinder $-\mu \leqslant z \leqslant \lambda,\ 0 \leqslant r \leqslant 1$ for $r = 1$, take in all problems the form

$$\tau_{r\varphi} = \tau_{rz} = 0, \qquad -\mu \leqslant z \leqslant \lambda$$
$$\sigma_r = 0, \qquad -\mu \leqslant z < 0; \quad u = 0, \qquad 0 \leqslant z \leqslant \lambda \tag{2.1}$$

The solution can be sought in the form of the series

$$u_\varkappa = \sum_{k=-\infty}^{\infty} \sum_{s=1}^{3} u_\varkappa{}^{ks} \tag{2.2}$$

determining the coefficients A_{ks} from the conditions of periodicity and symmetry at the end faces $z = -\mu$, $z = \lambda$ and setting $A_{03} = 0$.

Problem 3. The bands are placed periodically, the bending is accomplished only by two moments M, applied at $z = \pm\infty$. In this case both ends are planes of symmetry and the boundary conditions have the form

$$\tau_{\varphi z}(-\mu, r) = \tau_{rz}(-\mu, r) = 0, \qquad w(-\mu, r) = ar\cos\varphi \tag{2.3}$$
$$\tau_{\varphi z}(\lambda, r) = \tau_{rz}(\lambda, r) = w(\lambda, r) = 0 \tag{2.4}$$

where $2a$ is the angle between the axes of neighboring bands. We expand the integrals which occur in the functions $u_\varkappa^{ks}$ in residue series for $z = -\mu$ and $z = \lambda$ and we insert (1.14), (1.20) into (2.2). Interchanging the summation order in the double series and taking into account the equalities $f_\varkappa(-p, r) = -(-1)^\varkappa f_\varkappa(p, r)$ $(\varkappa = 1, 2, \ldots, 6)$, we obtain

$$u_\varkappa(-\mu, r) = \sum_{k=1}^{\infty} \sum_{s=1}^{3} f_\varkappa(b_{ks}, r)\left[A_{-ks}(-1)^{\varkappa+1} e^{b_{ks}\mu} + e^{-b_{ks}\mu} \sum_{n=-\infty}^{\infty} \sum_{q=1}^{3} A_{nq}\psi_{sq}{}^{kn}\right] + \sum_{n=-\infty}^{\infty} \sum_{q=1}^{3} A_{nq}\varphi_q{}^{\varkappa n} \tag{2.5}$$

$$u_\varkappa(\lambda, r) = \sum_{k=1}^{\infty} \sum_{s=1}^{3} f_\varkappa(a_{ks}, r)\left[A_{ks} e^{a_{ks}\lambda} + (-1)^\varkappa e^{-a_{ks}\lambda} \sum_{n=-\infty}^{\infty} \sum_{q=1}^{3} A_{nq}\chi_{sq}{}^{kn}\right] \tag{2.6}$$

Here

$$\psi_{sq}^{kn} = \frac{b_{nq}^2 b_{ks}^2 K_0 K^-(-b_{nq}) D_1(b_{nq})}{(b_{ks} + b_{nq}) K^+(b_{ks}) D_2{}^*(b_{ks})} \qquad (n \leqslant -1)$$

$$\psi_{sq}^{kn} = -\frac{b_{ks}^2 K^+(a_{nq}) D_2(a_{nq})}{a_{nq}^2 (b_{ks} - a_{nq}) K^+(b_{ks}) D_2{}^*(b_{ks})} \qquad (n \geqslant 1)$$

$$\psi_{sq}^{k0} = (-1)^{s+1} b_{ks}^{2-s}[K^+(b_{ks}) D_2{}^*(b_{ks})]^{-1} \tag{2.7}$$

$$\chi_{sq}^{kn} = \frac{b_{nq}^2 K^-(-b_{nq}) D_1(b_{nq})}{a_{ks}^2 (a_{ks} - b_{nq}) K^-(-a_{ks}) D_1{}^*(a_{ks})} \qquad (n \leqslant -1)$$

$$\chi_{sq}^{kn} = -\frac{K^+(a_{nq}) D_2(a_{nq})}{a_{nq}^2 a_{ks}^2 (a_{ks} + a_{nq}) K_0 K^-(-a_{ks}) D_1{}^*(a_{ks})} \qquad (n \geqslant 1)$$

$$\chi_{sq}^{k0} = [a_{ks}^{s+2} K_0 K^-(-a_{ks}) D_1^*(a_{ks})]^{-1}$$

$$\varphi_q^{1n} = \varphi_q^{2n} r^{-1}(\mu + \eta_1) = -\varphi_q^{3n} = b_{nq} K^-(-b_{nq}) D_1(b_{nq})(\mu + \eta_1) \qquad (n \leqslant -1)$$

$$\varphi_q^{1n} = \varphi_q^{2n} r^{-1}(\mu + \eta_1) = -\varphi_q^{3n} = K_0^{-1} a_{nq}^{-3} K^+(a_{nq}) D_2(a_{nq})(\mu + \eta_1) \qquad (n \geqslant 1)$$

$$\eta_n = \frac{\partial^n}{\partial p^n} K^+(p)\,|_{p=0}, \qquad \varphi_1^{20} = K_0^{-1}(\mu + \eta_1)\,r$$

$$\varphi_2^{10} = -\varphi_2^{30} = K_0^{-1}\,[{}^1/_6\mu^3 + {}^1/_2\mu^2\eta_1 + \mu(\eta_1^2 - {}^1/_2\eta_2) + \eta_3 - \eta_1\eta_2 + \eta_1^3]$$

$$\varphi_1^{10} = \varphi_2^{20} = -\varphi_1^{30} = \partial\varphi_2^{10} / \partial\mu$$

By virtue of the fact that the elements (1.20) are self-balancing and according to (1.15), we have $A_{01} = \pi^{-1} M$, $A_{02} = 0$. We insert the series (2.5) into the conditions (2.3). We multiply both sides of the equalities (2.3) for $\varkappa = 4,6,2$ by $f_\varkappa(b_{ks}, r)$, respectively, for $\varkappa = 1,3,5$ and we integrate with respect to r from 0 to 1. Adding the first two equalities and subtracting the third one, by virtue of the generalized orthogonality relation [6]

$$\int_0^1 [f_1(b_{ks}, r) f_4(b_{nq}, r) + f_3(b_{ks}, r) f_6(b_{nq}, r) - f_5(b_{ks}, r) f_2(b_{nq}, r)]\, r\, dr = 0 \qquad (2.8)$$

which holds for $b_{ks}^2 \neq b_{nq}^2$, we obtain the value of the rotation angle

$$2a = 2 \sum_{n=-\infty}^{\infty} \sum_{q=1}^{3} A_{nq}\varphi_q^{2n} r^{-1} \qquad (2.9)$$

and the system ($s = 1,2,3$; $k = 1,2,\ldots$)

$$-X_{-ks} + \sum_{q=1}^{3} e^{-b_{ks}\mu} \Big[\sum_{n=-1}^{-\infty} \psi_{sq}^{kn} e^{b_{nq}\mu} X_{nq} + \sum_{n=0}^{\infty} \psi_{sq}^{kn} e^{-a_{nq}\lambda} X_{nq} \Big] = 0 \qquad (2.10)$$

Here

$$X_{-ks} = A_{-ks} e^{b_{ks}\mu}, \qquad X_{ks} = A_{ks} e^{a_{ks}\lambda}, \qquad a_{0q} = 0,\ X_{0q} = A_{0q}$$

Performing the same operations with the series (2.6) and making use of the relation (2.8) which holds if we replace b_{ks} by a_{ks} [6], we obtain

$$X_{ks} + \sum_{q=1}^{3} e^{-a_{ks}\lambda} \Big[\sum_{n=-1}^{-\infty} \chi_{sq}^{kn} e^{b_{nq}\mu} X_{nq} + \sum_{n=0}^{\infty} \chi_{sq}^{kn} e^{-a_{nq}\lambda} X_{nq} \Big] = 0 \qquad (2.11)$$

According to the estimates (1.11), the double series of the system (2.10), (2.11) converge absolutely because of the exponential factors and the free terms of the system are bounded. Thus, (2.10), (2.11) is a normal Poincaré-Koch system [7]. Based on the orthogonality relation (2.8) and on the Kirchhoff uniqueness theorem, we can establish the existence and uniqueness of its normal solution.

P r o b l e m 4 is the limiting case of Problem 3 for $\mu = \infty$. From (2.10) it follows that $A_{ks} = 0$ for $k \leqslant -1$. The unknowns A_{ks} for $k \geqslant 1$ are obtained from the system (2.11), whose matrix elements for $n \leqslant -1$ are equal to zero.

P r o b l e m 5 is the limiting case of Problem 3 for $\lambda = \infty$. Obviously, $A_{ks} = 0$ for $k \geqslant 1$. The system (2.10) for the determination of A_{ks} for $k \leqslant -1$ can be simplified because of $e^{-a_{nq}\lambda} = 0$.

P r o b l e m 6. Loads are applied to the bands, statically equivalent to the bending moment $2M$, acting in the axial plane $\varphi = 0$ in one direction. The boundary conditions are

$$u = a\cos\varphi,\ v = a\sin\varphi,\ \sigma_z = 0,\ z = -\mu,\ 0 \leqslant r \leqslant 1 \qquad (2.12)$$

$$u = v = \sigma_z = 0, \quad z = \lambda,\ 0 \leqslant r \leqslant 1 \qquad (2.13)$$

In the cross section $z = -\mu$ the resultant vector of the tangential stresses is equal to $M(\lambda + \mu)^{-1}$, therefore it follows from (1.16) that

$$A_{01} = \pi^{-1}(\eta_1 + \mu)(\lambda + \mu)^{-1}M, \qquad A_{02} = \pi^{-1}(\lambda + \mu)^{-1} M$$

In order to compute the remaining A_{ks} and the magnitudes of the relative displacements of the neighboring bands, we insert (2.5) into (2.12). We multiply the equalities (2.12) for $\varkappa = 1,3,5$ by $f_\varkappa(b_{ks}, r)$ for $\varkappa = 4,6,2$, respectively. Integrating them with respect to r from 0 to 1 and adding them, we obtain by virtue of (2.8) ($s = 1,2,3$; $k = 1,2,\ldots$)

$$X_{-ks} + \sum_{q=1}^{3} e^{-b_{ks}\mu}\Big[\sum_{n=-1}^{-\infty} \psi_{sq}^{kn} e^{b_{nq}\mu} X_{nq} + \sum_{n=0}^{\infty} \psi_{sq}^{kn} e^{-a_{nq}\lambda} X_{nq}\Big] = 0 \qquad (2.14)$$

$$2a = 2\sum_{q=1}^{3}\sum_{n=-\infty}^{\infty} A_{nq}\varphi_q^{1n} \qquad (2.15)$$

In a similar way we obtain from the conditions (2.13)

$$X_{ks} - \sum_{q=1}^{3} e^{-a_{ks}\lambda}\Big[\sum_{n=-1}^{-\infty} \chi_{sq}^{kn} e^{b_{nq}\mu} X_{nq} + \sum_{n=0}^{\infty} \chi_{sq}^{kn} e^{-a_{nq}\lambda} X_{nq}\Big] = 0 \qquad (2.16)$$

The system (2.14), (2.16) is similar to (2.10), (2.11).

Problem 7 is a special case of the preceding one for sufficiently large μ, when the numbers $e^{-b_{ks}\mu}$ are so small that, taking into account (2.14), we can take $A_{ks} = 0$ for $k \leqslant -1$, while for $k \geqslant 1$ we can find A_{ks} from the system (2.16). As in Problem 2, the solution near the end of a semi-infinite cylinder will be approximate.

Problem 8 differs from Problem 6 by the fact that the given moments have alternating directions. Its boundary conditions are (2.3), (2.13), the system for A_{ks} is (2.10), (2.16), the rotation of neighboring bands is determined by formula (2.9), and $A_{01} = \pi^{-1} M$, $A_{02} = 0$.

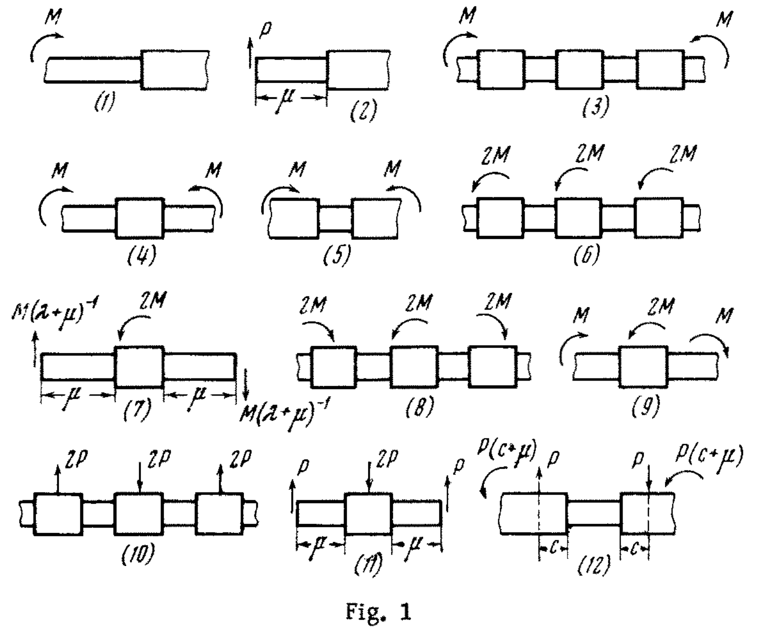

Fig. 1

Problem 9 is obtained from the preceding one when $\mu = \infty$. For $k \leqslant -1$ we have $A_{ks} = 0$, for $k \geqslant 1$ we obtain A_{ks} from the truncated system (2.16).

Problem 10. There are loads on the band, statically equivalent to the forces $2P$, applied at the middle of the bands along the radius in the plane $\varphi = 0$ and alternating in direction. Here the boundary conditions are (2.4), (2.12), the coefficients A_{ks} are determined from the system (2.11), (2.14), the coefficients A_{0s} from the formulas (1.16), and the shift of neighboring bands from the series (2.15).

Problem 11, similarly to Problems 2 and 7, can be solved for $\mu \gg 1$. Then it is a special case of the Problem 10 for $A_{ks} = 0$ for all $k \leqslant -1$. For $k \geqslant 1$, A_{ks} are determined from (2.11).

Problem 12 is a limiting case of Problem 10 for $\lambda = \infty$. Here for $k \geqslant 1$, $A_{ks} = 0$, for $k \leqslant -1$, A_{ks} are obtained from the truncated system (2.14) and the shift is obtained from (2.15).

By the superposition of the solutions of the above considered problems we can investigate the bending of cylinders under complicated conditions, for example when the bands rotate and move in different axial planes or when excentric forces are applied to them.

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